Quasi-periodic solutions for some (2 + 1)-dimensional integrable models generated by the Jaulent-Miodek hierarchy

2001 J. Phys. A: Math. Gen. 34 989

(http://iopscience.iop.org/0305-4470/34/5/305)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 989–1004 www.iop.org/Journals/ja PII: S0305-4470(01)14251-4

Quasi-periodic solutions for some (2 + 1)-dimensional integrable models generated by the Jaulent–Miodek hierarchy

Xianguo Geng[1,2], Cewen Cao[2] and H H Dai[3]

[1] CCAST (World Laboratory), PO Box 8730, Beijing 100080, People's Republic of China
[2] Department of Mathematics, Zhengzhou University, Zhengzhou, Henan 450052, People's Republic of China
[3] Department of Mathematics, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong, People's Republic of China

Received 24 May 2000, in final form 21 November 2000

Abstract
Some (2 + 1)-dimensional integrable models, including the modified Kadomtsev–Petviashvili equation, generated by the Jaulent–Miodek hierarchy are investigated. With the help of the Jaulent–Miodek eigenvalue problem, these (2 + 1)-dimensional integrable models are separated into compatible Hamiltonian systems of ordinary differential equations. Using the generating function flow method, the involutivity and the functional independence of the integrals are proved. The Abel–Jacobi coordinates are introduced, from which the quasi-periodic solutions for these (2+1)-dimensional integrable models are derived by resorting to the Riemann theta functions.

PACS number: 0230J

1. Introduction

The study of soliton hierarchies is one of the most prominent subjects in the field of nonlinear science. A fairly satisfactory understanding has been obtained for the (1 + 1)-dimensional integrable models over recent decades. Some important explicit solutions have been found, including the N-soliton solution, the quasi-periodic (or finite-band, or algebro-geometric) solution, and the polar expansion solution. Quite a few systematic methods have been developed, such as the inverse scattering transformation [1–3], the bilinear transformation methods of Hirota [4], the dressing method [3], the Bäcklund and the Darboux transformations [5, 6], the algebraic curve method [7], the nonlinearization approach of eigenvalue problems or Lax pairs [8–10], and so on. The situation is not so good for the (2 + 1)-dimensional integrable models, which are more complicated and more difficult. They refuse to yield to quite a few usual methods of analysis, though they are successful in the 1 + 1 case. Nevertheless, some progress has been made. For example, much has been done for the Kadomtsev–Peiviashvili (KP) equation [1, 3]

$$w_t = \tfrac{1}{4}(w_{xx} + 3w^2)_x + \tfrac{3}{4}\partial_x^{-1} w_{yy} \tag{1.1}$$

whose N-soliton solution, the quasi-periodic solution and some other explicit solutions are found through elaborate methods. Unfortunately, one usually encounters great difficulties in trying to extend these methods to other examples because of the strong characteristics of the $(2 + 1)$-dimensional models.

Based on the nonlinearization of the Lax pair and its adjoint, Konopelchenko *et al* [11] and Cheng and Li [12] found independently that some special solutions of the KP equation (1.1) can be obtained simply from the compatible solutions of the first two members of the Ablowitz, Kaup, Newell and Segur (AKNS) family, one of the well known $(1 + 1)$-dimensional soliton hierarchies. These two members are known as the coupled nonlinear Schrödinger equation,

$$u_y = u_{xx} - 2u^2 v \qquad v_y = -v_{xx} + 2uv^2 \tag{1.2}$$

and the coupled modified Korteweg–de Vries equation,

$$u_t = u_{xxx} - 6uu_x v \qquad v_t = v_{xxx} - 6uvv_x. \tag{1.3}$$

Specifically, let $(u(x, y, t), v(x, y, t))$ be a compatible solution of (1.2) and (1.3), then $w(x, y, t) = -2uv$ solves (1.1). This discovery seems to be one of the points favouring a breakthrough in tackling the difficult 2 + 1 models. Quite recently, we succeeded in calculating the well known algebro-geometric solution of the KP equation (1.1) [13]:

$$w(x, y, t) = 2\frac{\partial^2}{\partial x^2} \ln\theta(\Omega_1 x + \Omega_2 y + \Omega_3 t + D) + w_0 \tag{1.4}$$

using a newly designed scheme resorting to the above decomposition, which enables us to obtain the special quasi-periodic solutions of other 2 + 1 models such as the modified Kadomtsev–Peiviashvili (mKP) equation [14], the (2 + 1)-dimensional Caudrey–Dodd–Gibbon–Kotera–Sawada (2 + 1 CDGKS) equation [15], the (2 + 1)-dimensional modified Korteweg–de Vries (2 + 1 mKdV) equation [16], and a discrete model, the special 2 + 1 Toda equation [17]:

mKP:

$$w_t = \tfrac{1}{4}(w_{xx} - 2w^3)_x + \tfrac{3}{4}\left(\partial_x^{-1} w_{yy} - 2w_x \partial_x^{-1} w_y\right) \tag{1.5}$$

2 + 1 CDGKS:

$$\begin{aligned} w_t = &-\tfrac{1}{36}(w_{xxxxx} + 15ww_{xxx} + 15w_x w_{xx} + 45w^2 w_x) \\ &+ \tfrac{5}{36}\left(\partial_x^{-1} w_{yy} + 2w_x \partial_x^{-1} w_y + w_{xxy} + 3ww_y\right) \end{aligned} \tag{1.6}$$

2 + 1 mKdV:

$$w_t = \tfrac{1}{4}(w_{xx} - 2w^3)_x + \tfrac{3}{4}\left(\partial_x^{-1} w_{yy} - 2w_x \partial_x^{-1} w_y - 6ww_y\right) \tag{1.7}$$

2 + 1 special Toda:

$$\frac{\partial^2 w_n}{\partial x \partial y} = \exp(w_{n+1} - w_n)\frac{\partial}{\partial x}(w_{n+1} + w_n) - \exp(w_n - w_{n-1})\frac{\partial}{\partial x}(w_n + w_{n-1}). \tag{1.8}$$

Table 1.

2 + 1	1 + 1	U	
KP	AKNS	$\begin{pmatrix} \frac{1}{2}\lambda & u \\ v & -\frac{1}{2}\lambda \end{pmatrix}$	$w = -2uv$
mKP	Chen–Lee–Liu	$\begin{pmatrix} \frac{1}{2}(\lambda - uv) & \lambda u \\ v & -\frac{1}{2}(\lambda - uv) \end{pmatrix}$	$w = uv$
2 + 1 CDGKS	mKdV	$\begin{pmatrix} u & \lambda \\ 1 & -u \end{pmatrix}$	$w = -2u^2$
2 + 1 mKdV	Kaup–Newell	$\begin{pmatrix} -\frac{1}{2}\lambda & \lambda u \\ v & \frac{1}{2}\lambda \end{pmatrix}$	$w = uv$
2 + 1 special Toda	Toda	$\frac{1}{a}\begin{pmatrix} 0 & a^2 \\ -1 & \lambda - b \end{pmatrix}$	$w = \partial_x^{-1} b$

The special solution of each 2 + 1 model is constructed in a similar way by the compatible solution of two members of the associated (1 + 1)-dimensional soliton hierarchy, which is the isospectral class of some 2 × 2 eigenvalue problem

$$\chi_x = U\chi \tag{1.9}$$

or

$$E\chi = U\chi \tag{1.10}$$

in the discrete case, where E is the shift operator: $Ef(n) = f(n+1)$. We list the key elements in table 1.

As soon as we enter the domain of (1+1)-dimensional integrable models, quite a few ready tools come to our rescue. First, the special solutions of the (1 + 1)-dimensional equations are further decomposed into those of compatible Hamiltonian systems, which are obtained through the nonlinearization of the eigenvalue problem (1.9) or (1.10). Second, the algebraic curve method is introduced and the Abel–Jacobi coordinates are defined, by which various flows are straightened out so that they can be integrated by quadratures. Third, the Riemann–Jacobi inversion is used to yield the final expression of the explicit solution by means of the Riemann theta function, similar to (1.4). In short, our new scheme is composed of three steps:

(a) decomposition;
(b) straightening out;
(c) inversion.

Here, we are going to study the special quasi-periodic solutions of some (2 + 1)-dimensional integrable models generated by the Jaulent–Miodek hierarchy, a well known (1+1)-dimensional integrable model, associated with the eigenvalue problem [18]

$$\chi_x = U(\boldsymbol{u}, \lambda)\chi \qquad U(\boldsymbol{u}, \lambda) = \begin{pmatrix} 0 & 1 \\ u + \lambda v + \lambda^2 & 0 \end{pmatrix} \qquad \chi = \begin{pmatrix} \chi^{(1)} \\ \chi^{(2)} \end{pmatrix} \tag{1.11}$$

where u and v are two potentials, λ is a constant spectral parameter, $\boldsymbol{u} = (u, v)^T$. The key step is the following decomposition, which only depends on the two non-trivial soliton equations in the Jaulent–Miodek hierarchy, but does not depend on their Lax pairs.

Theorem 1. *Let* (u, v) *be a compatible solution of the first two Jaulent–Miodek equations:*

$$\begin{aligned} u_y &= -v_{xxx} + 4uv_x + 2u_x v \\ v_y &= -4u_x + 6vv_x \end{aligned} \tag{1.12}$$

and

$$\begin{aligned} u_t &= -u_{xxx} + 6uu_x - 6uvv_x - \tfrac{3}{2}u_x v^2 + \tfrac{9}{2}v_x v_{xx} + \tfrac{3}{2}vv_{xxx} \\ v_t &= -v_{xxx} + 6(uv)_x - \tfrac{15}{2}v^2 v_x. \end{aligned} \tag{1.13}$$

Then $w(x, y, t) = v(x, y, t)$ *solves any one* $(2+1)$*-dimensional equation in the following list:*

$$w_t = -\left(w_{xx} - 2w^3\right)_x - \tfrac{3}{2}\left(w_x \partial_x^{-1} w_y + ww_y\right) \tag{1.14}$$

$$w_t = \tfrac{1}{2}\left(w_{xx} - 2w^3\right)_x + \tfrac{3}{2}\left(-\tfrac{1}{4}\partial_x^{-1} w_{yy} + ww_y\right) \tag{1.15}$$

$$w_t = -\tfrac{1}{4}\left(w_{xx} - 2w^3\right)_x - \tfrac{3}{4}\left(\tfrac{1}{4}\partial_x^{-1} w_{yy} + w_x \partial_x^{-1} w_y\right) \tag{1.16}$$

$$w_t = 2\left(w_{xx} - 2w^3\right)_x - \tfrac{3}{4}\left(\partial_x^{-1} w_{yy} - 2w_x \partial_x^{-1} w_y - 6ww_y\right) \tag{1.17}$$

where ∂_x^{-1} *denotes an inverse operator of* $\partial_x = \partial/\partial x$ *with the condition* $\partial_x \partial_x^{-1} = \partial_x^{-1}\partial_x = 1$*, which can be defined as* $(\partial_x^{-1} f)(x) = \int_{-\infty}^{x} f(x')\,\mathrm{d}x'$ *under the decaying condition at infinity.*

Proof. From (1.12) we have

$$\begin{aligned} u &= \tfrac{3}{4}w^2 - \tfrac{1}{4}\partial_x^{-1} w_y \\ \tfrac{1}{4}\partial_x^{-1} w_{yy} &= \left(w_{xx} - 2w^3\right)_x + w_x \partial_x^{-1} w_y + 2ww_y. \end{aligned}$$

Substituting into the second equation of (1.13), we have (1.14). Equations (1.15)–(1.17) are obtained through elementary calculations. □

Note that (1.16) becomes the mKP equation (1.5) [19] after changing the scales: $t \to -t$, $y \to -\frac{1}{2}y$.

The Jaulent–Miodek eigenvalue problem (1.11) is a little more complicated to treat than those mentioned above. On the other hand, it can govern more $(2+1)$-dimensional equations such as (1.14)–(1.17). This is a good balance.

The outline of the present paper is as follows. In section 2, a class of $(2+1)$-dimensional nonlinear evolution equations is proposed and decomposed into the $(1+1)$-dimensional Jaulent–Miodek equations. In sections 3 and 4, the finite-dimensional Hamiltonian system related to the Jaulent–Miodek hierarchy and their involutive systems of conserved integrals are studied. The class of $(2+1)$-dimensional nonlinear evolution equations is further separated into compatible Hamiltonian systems of ordinary differential equations. In section 5, resorting to the elliptic and quasi-Abel–Jacobi coordinates, the independence of two involutive systems of conserved integrals is proved. In sections 6 and 7, the Abel–Jacobi coordinates are introduced to straighten out the associated flows. The Riemann–Jacobi inversion is discussed, from which the quasi-periodic solutions for the class of $(2+1)$-dimensional nonlinear evolution equations, including (1.14)–(1.17), are obtained by using the Riemann theta functions.

2. A class of (2 + 1)-dimensional evolution equations and their decomposition

In this section, we shall propose a class of (2 + 1)-dimensional nonlinear evolution equations and their decomposition. To this end, we introduce recursive equations for two functions u and v:

$$B_{-1} = 2 \qquad B_0 = -v$$

$$\begin{aligned} B_{m+1} = {} & \tfrac{1}{16}\sum_{j=0}^{m}\left(2B_{j-1}B_{m-1-j,xx} - 4uB_{j-1}B_{m-1-j} - B_{j-1,x}B_{m-1-j,x}\right) \\ & -\tfrac{1}{4}\sum_{j=0}^{m}B_jB_{m-j} - \tfrac{1}{4}v\sum_{j=0}^{m+1}B_{j-1}B_{m-j} \qquad m \geqslant 0. \end{aligned} \tag{2.1}$$

It is easy to see that B_m are uniquely determined by (2.1) and the first few members are

$$\begin{aligned} & B_1 = -u + \tfrac{3}{4}v^2 \qquad B_2 = \tfrac{1}{4}\left(-v_{xx} + 6uv - \tfrac{5}{2}v^3\right) \\ & B_3 = -\tfrac{1}{4}u_{xx} + \tfrac{3}{4}u^2 + \tfrac{5}{8}vv_{xx} + \tfrac{5}{16}v_x^2 - \tfrac{15}{8}uv^2 + \tfrac{35}{64}v^4. \end{aligned} \tag{2.2}$$

Let us consider the transformation

$$u = \tfrac{3}{4}w^2 - \tfrac{1}{4}\partial_x^{-1}w_y \qquad v = w \tag{2.3}$$

which defines a map

$$\boldsymbol{u} = f(w) \qquad \boldsymbol{u} = (u, v)^T. \tag{2.4}$$

Now we introduce a class of (2 + 1)-dimensional nonlinear evolution equations

$$\frac{\partial w}{\partial t_m} = \frac{\partial \mathcal{B}_m}{\partial x} \qquad m \geqslant 1 \tag{2.5}$$

with

$$\mathcal{B}_m = 4B_m(\boldsymbol{u})|_{\boldsymbol{u}=f(w)}.$$

The first three (2 + 1)-dimensional nonlinear evolution equations in (2.5) are as follows:

$$w_{t_1} = w_y$$

$$w_{t_2} = -w_{xxx} + 6w^2w_x - \tfrac{3}{2}ww_y - \tfrac{3}{2}w_x\partial_x^{-1}w_y \tag{2.6}$$

$$w_{t_3} = \tfrac{1}{4}w_{xxy} + ww_{xxx} + \tfrac{1}{2}w_xw_{xx} - 7w^3w_x + \tfrac{3}{4}w^2w_y + \tfrac{3}{8}w_y\partial_x^{-1}w_y + \tfrac{3}{2}ww_x\partial_x^{-1}w_y \tag{2.7}$$

where equation (2.6) is the same as (1.14) when $t_2 = t$.

In what follows, we first construct the hierarchy of Jaulent–Miodek equations. Then the decomposition of the (2 + 1)-dimensional evolution equations (2.5) is given. Differentiating (2.1) with respect to x, we have

$$\begin{aligned} 4\partial_x B_{m+1} = {} & \tfrac{1}{2}\sum_{j=0}^{m}B_{m-1-j}(\partial_x^3 - 2u\partial_x - 2\partial_x u)B_{j-1} \\ & -2\sum_{j=0}^{m}B_{m-j}\partial_x B_j - \sum_{j=0}^{m+1}B_{m-j}(\partial_x v + v\partial_x)B_{j-1} \\ = {} & 2B_{-1}\partial_x B_{m+1} - 2B_m\partial_x B_0 - B_m(\partial_x v + v\partial_x)B_{-1} \\ & + \tfrac{1}{2}\sum_{j=0}^{m}B_{m-1-j}[(\partial_x^3 - 2u\partial_x - 2\partial_x u)B_{j-1} - 2(\partial_x v + v\partial_x)B_j - 4\partial_x B_{j+1}] \end{aligned} \tag{2.8}$$

which implies

$$\sum_{j=0}^{m} B_{m-1-j}[(\partial_x^3 - 2u\partial_x - 2\partial_x u)B_{j-1} - 2(\partial_x v + v\partial_x)B_j - 4\partial_x B_{j+1}] = 0. \qquad (2.9)$$

By induction we have from (2.9) that

$$(\partial_x^3 - 2u\partial_x - 2\partial_x u)B_{j-1} - 2(\partial_x v + v\partial_x)B_j - 4\partial_x B_{j+1} = 0. \qquad (2.10)$$

Equation (2.10) can be written as

$$Kg_{j-1} = Jg_j \qquad Jg_{-1} = 0 \qquad g_j = (B_j, B_{j+1})^T \qquad j \geqslant 0 \qquad (2.11)$$

with two skew-symmetric operators

$$K = \begin{pmatrix} \partial_x^3 - 2u\partial_x - 2\partial_x u & 0 \\ 0 & 4\partial_x \end{pmatrix} \qquad J = \begin{pmatrix} 2(\partial_x v + v\partial_x) & 4\partial_x \\ 4\partial_x & 0 \end{pmatrix}.$$

It is easy to calculate that

$$g_{-1} = \begin{pmatrix} 2 \\ -v \end{pmatrix} \qquad g_0 = \begin{pmatrix} -v \\ -u + \frac{3}{4}v^2 \end{pmatrix} \qquad g_1 = \begin{pmatrix} -u + \frac{3}{4}v^2 \\ \frac{1}{4}(-v_{xx} + 6uv - \frac{5}{2}v^3) \end{pmatrix}.$$

Therefore, the Jaulent–Miodek hierarchy can be written as [20]

$$\boldsymbol{u}_{t_m} = X_m \qquad m \geqslant 0 \qquad (2.12)$$

with the Jaulent–Miodek vector field $X_j = Kg_{j-1} = Jg_j$. The first two non-trivial systems are (1.12) and (1.13) with $y = t_1, t = t_2$. Then it is easy to see that if (u, v) is a solution of (1.12) and (2.12) with $m \geqslant 2$, the function $w(x, y, t_m) = v$ determined by the second expression of (2.3) is a solution of the (2 + 1)-dimensional evolution equations (2.5).

Let us introduce the generating function of $\{g_k\}$:

$$g_\lambda = g_{-1} + \sum_{k=0}^{\infty} g_k \lambda^{-k-1} \qquad (2.13)$$

which satisfies

$$(K - \lambda J)g_\lambda = 0. \qquad (2.14)$$

Assume that

$$V(\boldsymbol{u}, \lambda) = \sigma(\boldsymbol{u}, \lambda)[\gamma] = \begin{pmatrix} \gamma_x^{(1)} & -2\gamma^{(1)} \\ \gamma_{xx}^{(1)} - 2(u + \lambda v + \lambda^2)\gamma^{(1)} & -\gamma_x^{(1)} \end{pmatrix} \qquad (2.15)$$

then we have

$$V_x - [U, V] = U'_*\{(K - \lambda J)\gamma\}. \qquad (2.16)$$

Resorting to (2.16), $(K - \lambda J)\gamma = 0$ implies $(\partial/\partial x) \det \sigma(\gamma) = 0$. Thus we obtain from (2.13) that

$$\det \sigma[g_\lambda] = -16\lambda^2. \qquad (2.17)$$

3. The finite-dimensional Hamiltonian systems

In this section, we shall discuss the finite-dimensional Hamiltonian systems associated with the eigenvalue problem (1.11) with the help of the nonlinearization approach. Assume that λ_k and $\chi = (p_k, q_k)^T$ $(1 \leqslant k \leqslant N)$, are N distinct eigenvalues and the associated eigenfunctions for the eigenvalue problem (1.11). Then we have

$$\sigma(\boldsymbol{u}, \lambda_k)[\nabla\lambda_k] = 2\begin{pmatrix} p_k q_k & -p_k^2 \\ q_k^2 & -p_k q_k \end{pmatrix} \equiv \varepsilon_k \tag{3.1}$$

$$\frac{\partial \varepsilon_k}{\partial x} = [U(\boldsymbol{u}, \lambda_k), \varepsilon_k] \tag{3.2}$$

$$(K - \lambda_k J)\nabla\lambda_k = 0 \tag{3.3}$$

where

$$\nabla\lambda_k = (p_k^2, \lambda_k p_k^2)^T. \tag{3.4}$$

We now introduce the Bargmann constraint

$$g_0 = \sum_{k=1}^{N} \nabla\lambda_k \tag{3.5}$$

or

$$\boldsymbol{u} = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{3}{4}\langle p, p\rangle^2 - \langle \Lambda p, p\rangle \\ -\langle p, p\rangle \end{pmatrix} \equiv h(p, q) \tag{3.6}$$

which plays a central role in the process of nonlinearization for the eigenvalue problem (1.11). The original motivation for this kind of constraint comes from the scattering expression of the reflectionless potential or the Bargmann potential [21], which was summarized in the general form of (3.5) in [8]. Indeed, the Bargmann constraint (3.5) is exactly the scattering expression of the reflectionless potentials for the eigenvalue problem (1.11) (see [22]). Constraint (3.6) nonlinearizes N copies of the eigenvalue problem (1.11)

$$\begin{pmatrix} p_k \\ q_k \end{pmatrix}_x = U(\boldsymbol{u}, \lambda_k)\begin{pmatrix} p_k \\ q_k \end{pmatrix} \qquad (k = 1, \ldots, N) \tag{3.7}$$

into a finite-dimensional Hamiltonian system

$$\begin{pmatrix} p \\ q \end{pmatrix}_x = I\nabla\mathcal{H}_0 = \begin{pmatrix} -\partial\mathcal{H}_0/\partial q \\ \partial\mathcal{H}_0/\partial p \end{pmatrix} \tag{3.8}$$

where $\langle\cdot,\cdot\rangle$ denotes the canonical inner product in $\mathbb{R}^N$, $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$, $p = (p_1, \ldots, p_N)^T$, $q = (q_1, \ldots, q_N)^T$,

$$\mathcal{H}_0 = -\tfrac{1}{2}\langle q, q\rangle + \tfrac{1}{8}\langle p, p\rangle^3 - \tfrac{1}{2}\langle p, p\rangle\langle \Lambda p, p\rangle + \tfrac{1}{2}\langle \Lambda^2 p, p\rangle.$$

Let $Q_\lambda(\xi, \eta) = \langle(\lambda I - \Lambda)^{-1}\xi, \eta\rangle$. Consider

$$G_\lambda = \begin{pmatrix} 2 \\ \langle p, p\rangle \end{pmatrix} + \sum_{j=1}^{N} \frac{\nabla\lambda_j}{\lambda - \lambda_j} = \begin{pmatrix} 2 + Q_\lambda(p, p) \\ \lambda Q_\lambda(p, p) \end{pmatrix} \tag{3.9}$$

which satisfies $(K - \lambda J)G_\lambda = 0$ by a direct calculation. Resorting to (2.15) and (3.8), we have

$$V_\lambda = \sigma(\lambda)[G_\lambda] = \begin{pmatrix} 0 & -4 \\ \alpha_\lambda & 0 \end{pmatrix} + 2\begin{pmatrix} Q_\lambda(p, q) & -Q_\lambda(p, p) \\ Q_\lambda(q, q) & -Q_\lambda(p, q) \end{pmatrix} \tag{3.10}$$

where

$$\alpha_\lambda = -\langle p, p\rangle^2 + 2\langle \Lambda p, p\rangle + 2\lambda\langle p, p\rangle - 4\lambda^2.$$

Here the following formulae are used:

$$Q_\lambda(\Lambda\xi, \eta) = \lambda Q_\lambda(\xi, \eta) - \langle \xi, \eta\rangle$$
$$Q_\lambda(\Lambda^2\xi, \eta) = \lambda^2 Q_\lambda(\xi, \eta) - \lambda\langle \xi, \eta\rangle - \langle \Lambda\xi, \eta\rangle.$$

Note that $F_\lambda = \det V_\lambda$ is invariant under the action of the x-flow. Based on the identity

$$Q_\lambda(\xi, \eta) = \sum_{m=0}^{\infty} \lambda^{-m-1}\langle \Lambda^m\xi, \eta\rangle$$

the generating function of integrals of motion for the finite-dimensional Hamiltonian system (3.8) can be written as

$$\begin{aligned} F_\lambda = \det V_\lambda &= (4 + 2Q_\lambda(p, p))(\alpha_\lambda + 2Q_\lambda(q, q)) - 4Q_\lambda^2(p, q) \\ &= -16\lambda^2 + \sum_{m=0}^{\infty} \lambda^{-m-1} F_m \end{aligned} \tag{3.11}$$

with

$$\begin{aligned} F_0 &= -16\mathcal{H}_0 \\ F_m &= -8\langle \Lambda^{m+2} p, p\rangle + 8\langle \Lambda^m q, q\rangle + 4\langle p, p\rangle\langle \Lambda^{m+1} p, p\rangle \\ &\quad + 2(-\langle p, p\rangle^2 + 2\langle \Lambda p, p\rangle)\langle \Lambda^m p, p\rangle \\ &\quad + 4\sum_{j=1}^{m}(\langle \Lambda^{j-1} p, p\rangle\langle \Lambda^{m-j} q, q\rangle - \langle \Lambda^{j-1} p, q\rangle\langle \Lambda^{m-j} p, q\rangle) \qquad m \geqslant 1. \end{aligned} \tag{3.12}$$

Consider F_λ as a Hamiltonian in the symplectic space $(\mathbb{R}^{2N}, \mathrm{d}p \wedge \mathrm{d}q)$. A direct calculation gives the canonical equations of the F_λ-flow:

$$\frac{\mathrm{d}}{\mathrm{d}\tau_\lambda}\begin{pmatrix} p_k \\ q_k \end{pmatrix} = I\nabla_k F_\lambda = \begin{pmatrix} -\partial F_\lambda/\partial q_k \\ \partial F_\lambda/\partial p_k \end{pmatrix} = W(\lambda, \lambda_k)\begin{pmatrix} p_k \\ q_k \end{pmatrix} \tag{3.13}$$

where

$$W(\lambda, \mu) = \frac{4}{\lambda - \mu} V_\lambda + \Delta_\lambda(\mu)\sigma_3 \qquad \sigma_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\Delta_\lambda(\mu) = -4(\lambda + \mu - \langle p, p\rangle)V_\lambda^{12} = 8(2 + Q_\lambda(p, p))(\lambda + \mu - \langle p, p\rangle).$$

As a consequence of (3.13) the matrix ε_k defined by (3.1) satisfies

$$\frac{\mathrm{d}\varepsilon_k}{\mathrm{d}\tau_\lambda} = [W(\lambda, \lambda_k), \varepsilon_k]. \tag{3.14}$$

Theorem 2. *The Lax matrix V_μ satisfies the Lax equation alone the τ_λ-flow:*

$$\frac{\mathrm{d}V_\mu}{\mathrm{d}\tau_\lambda} = [W(\lambda, \mu), V_\mu]. \tag{3.15}$$

Besides,

$$\{F_\mu, F_\lambda\} = 0 \qquad \forall\lambda \quad \mu \in \mathbb{C} \tag{3.16}$$

$$\{F_j, F_k\} = 0 \qquad \forall j \quad k = 0, 1, 2, \ldots. \tag{3.17}$$

Proof. Note the identities

$$Q_\lambda(\Lambda\xi,\eta) = \lambda Q_\lambda(\xi,\eta) - \langle\xi,\eta\rangle$$
$$\langle(\mu I-\Lambda)^{-1}(\lambda I-\Lambda)^{-1}\xi,\eta\rangle = \frac{1}{\mu-\lambda}(Q_\lambda(\xi,\eta) - Q_\mu(\xi,\eta)).$$

A direct calculation shows that (3.15) holds. As a consequence of the Lax equation (3.15) we obtain

$$0 = \frac{\mathrm{d}}{\mathrm{d}\tau_\lambda}\det V_\mu = \frac{\mathrm{d}F_\mu}{\mathrm{d}\tau_\lambda} = \{F_\mu, F_\lambda\}.$$

Substituting the expansion (3.11) into (3.16) gives (3.17) by comparing the same power of λ, μ. □

4. Other integrals $\{H_k\}$ and soliton equations

In order to establish the direct relation between finite-dimensional Hamiltonian systems and the Jaulent–Miodek field X_k, we define a new set of integrals $\{H_k\}$ recursively by

$$H_0 = \tfrac{1}{4}F_0 \qquad H_1 = \tfrac{1}{4}F_1 \qquad H_2 = \tfrac{1}{4}F_2$$
$$H_{m+3} = \tfrac{1}{4}F_{m+3} + \tfrac{1}{16}\sum_{\substack{i+j=m\\ i,j\geqslant 0}} H_i H_j \qquad m = 0, 1, 2, \ldots \tag{4.1}$$

which is put in the equivalent form

$$-\frac{1}{16\lambda^2}F_\lambda = (1 - \tfrac{1}{8}H_\lambda)^2 \tag{4.2}$$

with the help of the generating function

$$H_\lambda = \sum_{k=0}^{\infty} H_k \lambda^{-k-3}. \tag{4.3}$$

The involutivity of $\{H_k\}$ is based on the equality

$$\{H_\mu, H_\lambda\} = \frac{1}{\lambda\mu\sqrt{F_\lambda F_\mu}}\{F_\mu, F_\lambda\} = 0.$$

Acting with $J^{-1}K$ upon the Bargmann constraint (3.5) k times and noting that $\ker J = \{\varrho_1 g_{-1} + \varrho_2 g_{-2} | g_{-2} = (0,1)^T, \forall\varrho_1, \varrho_2\}$ gives

$$\sum_{j=1}^{N} \lambda_j^k \nabla\lambda_j = g_k + c_2 g_{k-2} + c_3 g_{k-3} + \cdots + c_{k+1} g_{-1} + \zeta_{k+2} g_{-2} \qquad k \geqslant 1 \tag{4.4}$$

in view of (2.11) and (3.3), where c_j and ζ_{k+2} are constants of integration. Resorting to the structures of g_0 and g_1, we find that $c_2 = 0$ as $k = 1$. By using (3.9) and (4.4), we arrive at

$$G_\lambda = g_{-1} + \sum_{j=1}^{N} \frac{\nabla\lambda_j}{\lambda-\lambda_j} = g_{-1} + \sum_{k=0}^{\infty}\lambda^{-k-1}\sum_{j=1}^{N}\lambda_j^k\nabla\lambda_j$$
$$= c_\lambda g_\lambda + \sum_{k=1}^{\infty}\lambda^{-k-1}\zeta_{k+2} g_{-2} \tag{4.5}$$

with

$$c_\lambda = 1 + \sum_{k=0}^{\infty} c_{k+3}\lambda^{-k-3}. \tag{4.6}$$

Using (3.10) and (2.15), we have

$$V_\lambda = \sigma(\lambda)\left[c_\lambda g_\lambda + \sum_{k=1}^{\infty}\lambda^{-k-1}\zeta_{k+2}g_{-2}\right] = \sigma(\lambda)\,[c_\lambda g_\lambda] \tag{4.7}$$

$$F_\lambda = -16\lambda^2 c_\lambda^2 \tag{4.8}$$

because of (2.17). By comparing (4.2) and (4.8) we obtain

$$c_\lambda = 1 - \tfrac{1}{8}H_\lambda \qquad c_{k+3} = -\tfrac{1}{8}H_k \qquad (k = 0, 1, 2, \ldots). \tag{4.9}$$

Denote the variables of the H_λ-flow and H_k-flow by t_λ and t_k, respectively. By the Leibniz rule of the Poisson bracket we obtain

$$\frac{1}{4\lambda^2}\{\psi, F_\lambda\} = \left(1 - \tfrac{1}{8}H_\lambda\right)\{\psi, H_\lambda\}$$

for any smooth function ψ due to (4.2). Thus

$$\frac{\mathrm{d}}{\mathrm{d}t_\lambda} = \frac{1}{4\lambda^2(1 - \frac{1}{8}H_\lambda)}\frac{\mathrm{d}}{\mathrm{d}\tau_\lambda} = \frac{1}{4\lambda^2 c_\lambda}\frac{\mathrm{d}}{\mathrm{d}\tau_\lambda}. \tag{4.10}$$

For $\boldsymbol{u} = h(p, q)$, we have

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}\tau_\lambda} = \begin{pmatrix} 3\langle p, p\rangle\left\langle p, \frac{\mathrm{d}p}{\mathrm{d}\tau_\lambda}\right\rangle - 2\left\langle \Lambda p, \frac{\mathrm{d}p}{\mathrm{d}\tau_\lambda}\right\rangle \\ -2\left\langle p, \frac{\mathrm{d}p}{\mathrm{d}\tau_\lambda}\right\rangle \end{pmatrix} = 4JG_\lambda \tag{4.11}$$

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t_\lambda} = \frac{1}{4\lambda^2 c_\lambda}\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}\tau_\lambda} = \frac{1}{\lambda^2 c_\lambda}JG_\lambda = \frac{1}{\lambda^2}Jg_\lambda = \sum_{k=0}^{\infty} Jg_k\lambda^{-k-3} = \sum_{k=0}^{\infty} X_k\lambda^{-k-3} \tag{4.12}$$

by (3.13), (3.8), (3.9), (4.5) and (2.13). On the other hand, we have

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t_\lambda} = h_*\left\{\frac{\mathrm{d}}{\mathrm{d}t_\lambda}\begin{pmatrix} p \\ q \end{pmatrix}\right\} = h_*(I\nabla H_\lambda) = \sum_{k=0}^{\infty} h_*(I\nabla H_k)\lambda^{-k-3}. \tag{4.13}$$

Thus we obtain the following assertions.

Theorem 3.

$$h_*(I\nabla H_k) = X_k. \tag{4.14}$$

Theorem 4. *Let $(p(x, t_k), q(x, t_k))^T$ be a compatible solution of the $\mathcal{H}_0$- and H_k-flow. Then $\boldsymbol{u}(x, t_k) = h(p, q)$ solves the kth Jaulent–Miodek equation*

$$\boldsymbol{u}_{t_k} = X_k(\boldsymbol{u}). \tag{4.15}$$

Theorem 5. *Let $(p(x, y, t_m), q(x, y, t_m))^T$ be a compatible solution of the $\mathcal{H}_0$-, H_1-, H_m-flow ($m \geqslant 2$)*

$$\begin{pmatrix} p \\ q \end{pmatrix}_x = I\nabla\mathcal{H}_0 \qquad \begin{pmatrix} p \\ q \end{pmatrix}_y = I\nabla H_1 \qquad \begin{pmatrix} p \\ q \end{pmatrix}_{t_m} = I\nabla H_m.$$

Then $w(x, y, t_m) = -\langle p, p\rangle$ solves the $(2 + 1)$-dimensional evolution equations (2.5). Especially, for $m = 2$ and $t = t_2$, $w(x, y, t) = -\langle p, p\rangle$ solves the $(2 + 1)$-dimensional evolution equations (1.14)–(1.17).

Proof. Let $h_0^{t_0}$ and $h_m^{t_m}$ be the solution operators of the initial-value problems of the Hamiltonian systems $(\mathbb{R}^{2N}, \mathrm{d}p\wedge\mathrm{d}q, \mathcal{H}_0)$ and $(\mathbb{R}^{2N}, \mathrm{d}p\wedge\mathrm{d}q, H_m)$, respectively. Then $h_j^{t_j}$ and $h_m^{t_m}$ commute [23]. Put the solution in two ways

$$\begin{pmatrix} p \\ q \end{pmatrix} = h_0^x h_1^y \left\{ h_m^{t_m} \begin{pmatrix} p_0 \\ q_0 \end{pmatrix} \right\} = h_0^x h_m^{t_m} \left\{ h_1^y \begin{pmatrix} p_0 \\ q_0 \end{pmatrix} \right\}.$$

We see that $\boldsymbol{u} = h(p, q)$ solves $u_y = X_1, u_{t_m} = X_m$ simultaneously. Therefore, the function $w(x, y, t_m) = -\langle p, p\rangle$ solves (2.5). □

5. Elliptic coordinates and functional independence of integrals

It is easy to see that each one of F_λ, V_λ^{12} and V_λ^{21} as a rational function of λ, has simple poles at λ_js $(1 \leqslant j \leqslant N)$ since the coefficients of $(\lambda - \lambda_j)^2$ is zero in F_λ. We have

$$\begin{aligned} F_\lambda &= -V_\lambda^{12}V_\lambda^{21} - (V_\lambda^{11})^2 \\ &= -4Q_\lambda^2(p, q) + 4Q_\lambda(p, p)Q_\lambda(q, q) + 8Q_\lambda(q, q) + 2\alpha_\lambda Q_\lambda(p, p) + 4\alpha_\lambda \\ &= -16\frac{b(\lambda)}{a(\lambda)} = -16\frac{R(\lambda)}{a^2(\lambda)} \end{aligned} \tag{5.1}$$

$$V_\lambda^{12} = -4 - 2Q_\lambda(p, p) = -4\frac{n(\lambda)}{a(\lambda)} \tag{5.2}$$

where

$$\begin{aligned} a(\lambda) &= \prod_{k=1}^{N}(\lambda - \lambda_k) \qquad b(\lambda) = \prod_{k=1}^{N+2}(\lambda - \lambda_{k+N}) \\ R(\lambda) &= a(\lambda)b(\lambda) = \prod_{j=1}^{2N+2}(\lambda - \lambda_j) \qquad n(\lambda) = \prod_{j=1}^{N}(\lambda - \mu_j). \end{aligned} \tag{5.3}$$

From (5.2) and (5.1) we have

$$w = v = -\langle p, p\rangle = 2\sum_{j=1}^{N}(\mu_j - \lambda_j) \tag{5.4}$$

$$V_{\mu_k}^{11} = 4\frac{\sqrt{R(\mu_k)}}{a(\mu_k)}. \tag{5.5}$$

Substitute $\mu = \mu_k$ in the 12th component of (4.15):

$$\frac{\mathrm{d}}{\mathrm{d}\tau_\lambda}V_\mu^{12} = \frac{8}{\lambda - \mu}(V_\lambda^{11}V_\mu^{12} - V_\mu^{11}V_\lambda^{12}). \tag{5.6}$$

After some calculations we obtain from (5.6) and (5.2) that

$$\begin{aligned} &\frac{1}{32\sqrt{R(\mu_k)}}\frac{\mathrm{d}\mu_k}{\mathrm{d}\tau_\lambda} = \frac{n(\lambda)}{a(\lambda)(\lambda - \mu_k)n'(\mu_k)} \\ &\sum_{k=1}^{N}\frac{\mu_k^{N-j}}{32\sqrt{R(\mu_k)}}\frac{\mathrm{d}\mu_k}{\mathrm{d}\tau_\lambda} = \frac{\lambda^{N-j}}{a(\lambda)} \qquad (j = 1, 2, \ldots, N) \end{aligned} \tag{5.7}$$

where the interpolation formula for polynomials is used. μ_k is called the elliptic coordinate of the finite-dimensional Hamiltonian system (3.8). Equation (5.7) suggests the consideration of the hyperelliptic curve Γ, defined by the affine equation

$$\xi^2 - (32)^2 R(\lambda) = 0 \tag{5.8}$$

with genus $g = N$ and the usual holomorphic differentials

$$\tilde{\omega}_j = \frac{\lambda^{N-j}\,\mathrm{d}\lambda}{32\sqrt{R(\lambda)}} \qquad j = 1, \ldots, N. \tag{5.9}$$

Denote $\rho(\mu_k) = \big(\lambda = \mu_k, \xi = 32\sqrt{R(\mu_k)}\big) \in \Gamma$. Let $P_0 \in \Gamma$ be fixed. Define the quasi-Abel–Jacobi coordinates by

$$\tilde{\phi}_j = \sum_{k=1}^{N} \int_{P_0}^{\rho(\mu_k)} \tilde{\omega}_j \qquad j = 1, \ldots, N. \tag{5.10}$$

Then the second expression of (5.7) is put in the form

$$\frac{\mathrm{d}\tilde{\phi}_j}{\mathrm{d}\tau_\lambda} = \frac{\lambda^{N-j}}{a(\lambda)}. \tag{5.11}$$

It is easy to prove that the coefficient of the expansion [13]

$$\frac{1}{(1-\lambda_1\lambda^{-1})\cdots(1-\lambda_N\lambda^{-1})} = \sum_{k=0}^{\infty} A_k \lambda^{-k} \tag{5.12}$$

can be determined recursively by

$$A_0 = 1 \qquad A_1 = s_1$$
$$A_k = \frac{1}{k}\left(s_k + \sum_{\substack{i+j=k \\ i,j\geqslant 1}} s_i A_j\right) \tag{5.13}$$

where $s_k = \lambda_1^k + \cdots + \lambda_N^k$. Denote the variable of F_k-flow by τ_k. Comparing the coefficients of λ^{-k-1} in the expansion of (5.11) gives

$$\frac{\mathrm{d}\tilde{\phi}_j}{\mathrm{d}\tau_k} = \{\tilde{\phi}_j, F_k\} = A_{k-j+1} \tag{5.14}$$

with supplementary definition $A_{-k} = 0 (k = 1, 2, \ldots)$. Thus

$$\left(\frac{\mathrm{d}\tilde{\phi}}{\mathrm{d}\tau_0}, \frac{\mathrm{d}\tilde{\phi}}{\mathrm{d}\tau_1}, \ldots, \frac{\mathrm{d}\tilde{\phi}}{\mathrm{d}\tau_{N-1}}\right) = \begin{pmatrix} 1 & A_1 & A_2 & \cdots & A_{N-1} \\ & 1 & A_1 & \cdots & A_{N-2} \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & A_1 \\ & & & & 1 \end{pmatrix} \tag{5.15}$$

where $\tilde{\phi} = (\tilde{\phi}_1, \ldots, \tilde{\phi}_N)^T$.

Theorem 6.

(a) $\{F_0, F_1, \ldots, F_{N-1}\}$ *are functionally independent;*
(b) $\{H_0, H_1, \ldots, H_{N-1}\}$ *are functionally independent.*

Proof.

(a) By [23], we need only prove the linear independence of the differentials $\mathrm{d}F_0, \mathrm{d}F_1, \ldots, \mathrm{d}F_{N-1}$. Suppose $\sum_{k=0}^{N-1} \gamma_k\, \mathrm{d}F_k = 0$. Let $H = \tilde{\phi}_j$ in the formula [23]

$$\{H, F\} = \omega^2(I\,\mathrm{d}F, I\,\mathrm{d}H).$$

We have

$$\sum_{k=0}^{N-1} \gamma_k\{\tilde{\phi}_j, F_k\} = 0 \qquad j = 1, \ldots, N.$$

Thus $\gamma_0 = \cdots = \gamma_{N-1} = 0$ because of (5.15).

(b) Noticing (4.1) and (4.9) we obtain

$$\begin{pmatrix} \mathrm{d}F_0 \\ \mathrm{d}F_1 \\ \mathrm{d}F_2 \\ \vdots \\ \mathrm{d}F_{N-1} \end{pmatrix} = 4 \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ & 1 & 0 & \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ & * & & \ddots & 0 \\ & & & & 1 \end{pmatrix} \begin{pmatrix} \mathrm{d}H_0 \\ \mathrm{d}H_1 \\ \mathrm{d}H_2 \\ \vdots \\ \mathrm{d}H_{N-1} \end{pmatrix}. \tag{5.16}$$

Thus $\mathrm{d}H_0, \ldots, \mathrm{d}H_{N-1}$ are also linearly independent. □

Corollary. *The finite-dimensional Hamiltonian system (3.8) is completely integrable in the Liouville sense.*

6. Abel–Jacobi coordinates

Let $a_1, b_1, \ldots, a_N, b_N$ be the canonical basis of cycles on the hyperelliptic curve Γ, and

$$C = (A_{jk})^{-1}_{N\times N} \qquad A_{jk} = \int_{a_k} \tilde{\omega}_j. \tag{6.1}$$

Define the normalized holomorphic differential by

$$\omega_s = \sum_{j=1}^{N} C_{sj}\tilde{\omega}_j \qquad \omega = (\omega_1, \ldots, \omega_N)^T = C\tilde{\omega}. \tag{6.2}$$

Then

$$\int_{a_k} \omega_s = \delta_{sk} \qquad \int_{b_k} \omega_s = B_{sk} \tag{6.3}$$

where the matrix $B = (B_{sk})$ is symmetric with positive-definite imaginary part and is used to define the Riemannian theta function of Γ [24, 25]:

$$\theta(\zeta) = \sum_{z\in\mathbb{Z}^N} \exp \pi\sqrt{-1}(\langle Bz, z\rangle + 2\langle \zeta, z\rangle) \qquad \zeta \in \mathbb{C}^N.$$

The Abel map $\mathcal{A}(P)$ and the Abel–Jacobi coordinates are defined as

$$\mathcal{A}(P) = \int_{P_0}^{P} \omega \qquad \mathcal{A}\left(\sum n_k P_k\right) = \sum n_k \mathcal{A}(P_k) \tag{6.4}$$

$$\phi = \mathcal{A}\left(\sum_{k=1}^{N} \rho(\mu_k)\right) = \sum_{k=1}^{N} \int_{P_0}^{\rho(\mu_k)} \omega = C\tilde{\phi}. \tag{6.5}$$

Let $S_k = \lambda_1^k + \cdots + \lambda_{2N+2}^k$ and $\hat{R}(\lambda^{-1}) = \prod_{j=1}^{2N+2}(1 - \lambda_j\lambda^{-1})$. Then the coefficients in (see [13])

$$\frac{1}{\sqrt{\hat{R}(\lambda^{-1})}} = \sum_{k=0}^{\infty} \Lambda_k \lambda^{-k} \tag{6.6}$$

are determined recursively by

$$\begin{aligned} &\Lambda_0 = 1 \qquad \Lambda_1 = \tfrac{1}{2} S_1 \\ &\Lambda_k = \frac{1}{2k}\left(S_k + \sum_{\substack{i+j=k \\ i,j\geqslant 1}} S_i \Lambda_j\right). \end{aligned} \tag{6.7}$$

From (5.1) and (4.2) we obtain

$$\lambda a(\lambda)(1 - \tfrac{1}{8}H_\lambda) = \sqrt{R(\lambda)}. \tag{6.8}$$

Let C_k be the kth column vector of the matrix C. By (6.5), (5.11), (4.10), (6.8) and the lemma we have

$$\begin{aligned} \frac{\mathrm{d}\phi}{\mathrm{d}\tau_\lambda} &= C\frac{\mathrm{d}\tilde{\phi}}{\mathrm{d}\tau_\lambda} = \frac{\lambda^N}{a(\lambda)}(C_1\lambda^{-1} + \cdots + C_N\lambda^{-N}) \\ \frac{\mathrm{d}\phi}{\mathrm{d}t_\lambda} &= \frac{1}{4\lambda^2(1 - \frac{1}{8}H_\lambda)}\frac{\mathrm{d}\phi}{\mathrm{d}\tau_\lambda} = \frac{\lambda^{N-1}}{4\sqrt{R(\lambda)}}(C_1\lambda^{-1} + \cdots + C_N\lambda^{-N}) \\ &= \frac{1}{4\lambda^2\sqrt{\hat{R}(\lambda^{-1})}}(C_1\lambda^{-1} + \cdots + C_N\lambda^{-N}) \\ &= \frac{1}{4}\sum_{k=0}^{\infty}\Lambda_k\lambda^{-k-2}\sum_{j=1}^{N}C_j\lambda^{-j} = \sum_{k=0}^{\infty}\Omega_k\lambda^{-k-3} \end{aligned} \tag{6.9}$$

with the constants

$$\begin{aligned} &\Omega_0 = \tfrac{1}{4}\Lambda_0 C_1 \qquad \Omega_1 = \tfrac{1}{4}(\Lambda_1 C_1 + \Lambda_0 C_2) \\ &\Omega_k = \tfrac{1}{4}(\Lambda_k C_1 + \cdots + \Lambda_1 C_k + \Lambda_0 C_{k+1}) \qquad k \leqslant N-1 \\ &\Omega_k = \tfrac{1}{4}(\Lambda_k C_1 + \cdots + \Lambda_{k-N+1} C_N) \qquad N \leqslant k. \end{aligned} \tag{6.10}$$

Therefore, we have the following fact.

Theorem 7. *Straightening out of the flow*

$$\frac{\mathrm{d}\phi}{\mathrm{d}t_\lambda} = \sum_{k=0}^{\infty}\Omega_k\lambda^{-k-3} \tag{6.11}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}t_k} = \Omega_k \qquad (k = 0, 1, 2, \ldots). \tag{6.12}$$

Note. From theorem 7, we arrive at the evolution picture of the 2 + 1 flows

$$\phi = \phi_0 + \Omega_0 x + \Omega_1 y + \Omega_k t_k \qquad k \geqslant 2. \tag{6.13}$$

7. The Riemann–Jacobi inversion and explicit solutions

Since deg $R = 2N+2$, on Γ there are two infinite points ∞_1 and ∞_2, which are not branch points of Γ. According to the Riemann theorem [24, 25], there exists a constant vector M (the Riemann constant) $\in \mathbb{C}^N$ such that $\theta(\mathcal{A}(\rho(\lambda)) - \phi - M)$ has exactly N zeros at $\lambda = \mu_1, \ldots, \mu_N$. And we have the inversion formula

$$\sum_{j=1}^{N} \mu_j = I_1(\Gamma) - \sum_{s=1}^{2} \operatorname*{Res}_{\lambda=\infty_s} \lambda \, \mathrm{d} \ln \theta(\mathcal{A}(\rho(\lambda)) - \phi - M) \tag{7.1}$$

with the constant

$$I_1(\Gamma) = \sum_{j=1}^{N} \int_{a_j} \lambda \omega_j .$$

For the same λ, there are two points on different sheets of the Riemann surface Γ:

$$\rho(\lambda) = \left(\lambda, 32\sqrt{R(\lambda)}\right) \qquad \rho_-(\lambda) = \left(\lambda, -32\sqrt{R(\lambda)}\right).$$

Under the local coordinate $z = \lambda^{-1}$ at infinity, the hyperelliptic curve Γ, $\xi^2 - (32)^2 R(\lambda) = 0$, in the neighbourhood of infinity is expressed as $\hat{\xi}^2 - (32)^2 \hat{R}(z) = 0$ with $\hat{\xi} = z^{N+1}\xi$, and $\left(z, 32(-1)^{s-1}\sqrt{R(z)}\right)_{z=0} = \left(0, 32(-1)^{s-1}\right)$, $s = 1, 2$. Then we have

$$\lambda^{-(N+1)}\sqrt{R(\lambda)} = (-1)^{-s-1}\sqrt{\hat{R}(z)}.$$

By (6.2) and (5.9) we obtain

$$\omega = C\tilde{\omega} = \frac{(-1)^s \, \mathrm{d}z}{32\sqrt{\hat{R}(z)}}(C_1 + zC_2 + \cdots + z^{N-1}C_N). \tag{7.2}$$

With the help of (6.9) and (6.4) we obtain

$$\omega = \tfrac{1}{8}(-1)^s \sum_{k=0}^{\infty} \Omega_k z^k \, \mathrm{d}z \tag{7.3}$$

$$\mathcal{A}(\rho(z^{-1})) = -\eta_s - \tfrac{1}{8}(-1)^{s-1} \sum_{k=0}^{\infty} \frac{1}{k+1} \Omega_k z^{k+1} \tag{7.4}$$

with

$$\eta_s = \int_{\infty_s}^{P_0} \omega .$$

Since the theta function is an even function, then we have

$$\theta(\mathcal{A}(\rho(z^{-1})) - \phi - M) = \theta(\phi + M + \eta_s) + \tfrac{1}{8} z (-1)^{s-1} \frac{\partial}{\partial x} \theta(\phi + M + \eta_s) + \mathrm{O}(z^2). \tag{7.5}$$

From (7.1) and (7.5) we arrive at

$$\begin{aligned} \sum_{j=1}^{N} \mu_j &= I_1 - \frac{1}{8} \sum_{s=1}^{2} (-1)^{s-1} \frac{\partial}{\partial x} \ln \theta(\phi + M + \eta_s) \\ &= I_1 + \frac{1}{8} \frac{\partial}{\partial x} \ln \frac{\theta(\phi + M + \eta_2)}{\theta(\phi + M + \eta_1)}. \end{aligned} \tag{7.6}$$

Therefore, we obtain by (5.4) and (6.13) that quasi-periodic solutions for the (2+1)-dimensional evolution equations (2.5)

$$w(x, y, t_m) = \frac{1}{4}\frac{\partial}{\partial x} \ln \frac{\theta(\Omega_0 x + \Omega_1 y + \Omega_m t_m + \Upsilon_2)}{\theta(\Omega_0 x + \Omega_1 y + \Omega_m t_m + \Upsilon_1)} + \varrho_0 \qquad m \geqslant 2 \tag{7.7}$$

with the constants

$$\Upsilon_1 = \phi_0 + M + \eta_1 \qquad \Upsilon_2 = \phi_0 + M + \eta_2 \qquad \varrho_0 = 2I_1 - 2\sum_{j=1}^{N} \lambda_j.$$

For $m = 2, t_2 = t$ the function (7.7) is the solution of the (2 + 1)-dimensional evolution equations (1.14)–(1.17).

Acknowledgments

This work was supported by Research Project of Nonlinear Science and National Natural Science Foundation of China (project no 10071075). One of the authors (H-HD) acknowledges the support by a grant from City University of Hong Kong (project no 7001072).

References

[1] Ablowitz M J and Clarkson P A 1991 *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge: Cambridge University Press)
[2] Newell A C 1981 *Solitons in Mathematics and Physics* (Philadelphia, PA: SIAM)
[3] Novikov S P, Manakov S V, Pitaevskii L P and Zakharov V E 1984 *Theory of Solitons, The Inverse Scattering Methods* (New York: Consultants Bureau)
[4] Matsuno Y 1984 *Bilinear Transformation Method* (New York: Academic)
[5] Lamb G L 1980 *Elements of Soliton Theory* (New York: Wiley)
[6] Matveev V B and Salle M A 1991 *Darboux Transformations and Solitons* (Berlin: Springer)
[7] Belokolos E D, Bobenko A I, Enol'skii V Z, Its A R and Matveev V B 1994 *Algebro-Geometric Approach to Nonlinear Integrable Equations* (Berlin: Springer)
[8] Cao C W and Geng X G 1990 *Nonlinear Physics (Research Reports in Physics)* (Berlin: Springer) pp 68–78
[9] Cao C W 1990 *Sci. China* A **33** 528
[10] Cao C W and Geng X G 1990 *J. Phys. A: Math. Gen.* **23** 4117
[11] Konopelchenko B, Sidorenko J and Strampp W 1991 *Phys. Lett.* A **157** 17
[12] Cheng Y and Li Y S 1991 *Phys. Lett.* A **157** 22
[13] Cao C W, Wu Y T and Geng X G 1999 *J. Math. Phys.* **40** 3948
[14] Geng X G, Wu Y T and Cao C W 1999 *J. Phys. A: Math. Gen.* **32** 3733
[15] Cao C W, Wu Y T and Geng X G 1999 *Phys. Lett.* A **256** 59
[16] Geng X G and Cao C W 1999 *Phys. Lett.* A **261** 289
[17] Cao C W, Geng X G and Wu Y T 1999 *J. Phys. A: Math. Gen.* **32** 8059
[18] Jaulent M and Miodek I 1976 *Lett. Math. Phys.* **1** 243
[19] Konopelchenko B and Dubrovsky V 1984 *Phys. Lett.* A **102** 45
[20] Zhou R G 1997 *J. Math. Phys.* **38** 1535
[21] Cao C W 1991 *Acta Math. Sin.* **7** 216
[22] Geng X G 1991 *Chin J. Contemp. Math.* **12** 1
[23] Arnold V I 1978 *Mathematical Methods of Classical Mechanics* (Berlin: Springer)
[24] Griffiths P and Harris J 1978 *Principles of Algebraic Geometry* (New York: Wiley)
[25] Mumford D 1984 *Tata Lectures on Theta II* (Boston, MA: Birkhäuser)